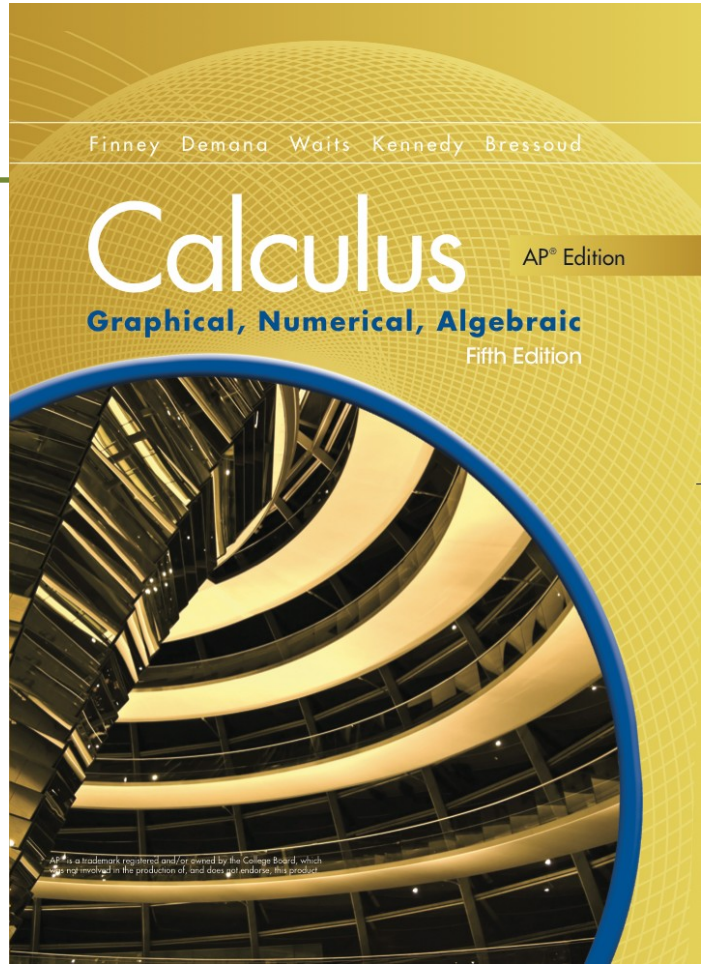
Chapter 4
More Derivatives

Section 4.4

Derivatives of Exponential and Logarithmic Functions



Copyright © 2015, 2012, and 2009 Pearson Education, Inc.

Quick Review

1. Write $\log_5 8$ in terms of natural logarithms.
2. Write 7^x as a power of e.

In Exercises 3 – 7, simplify the expression using properties of exponents and logarithms.

3. $\ln\left(e^{\tan x}\right)$
4. $\ln\left(x^2 - 4\right) - \ln\left(x + 2\right)$
5. $\log_2\left(8^{x-5}\right)$
6. $\dfrac{\left(\log_4 x^{15}\right)}{\left(\log_4 x^{12}\right)}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Review

7. $3\ln x - \ln 3x + \ln(12x^2)$

In Exercises 8 – 10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $\left(\frac{\ln 2}{3}\right)$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$

9. $5^x \ln 5 = 2^x$

10. $3^{x+1} = 2^x$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Review Solutions

1. Write $\log_5 8$ in terms of natural logarithms. $\dfrac{\ln 8}{\ln 5}$

2. Write 7^x as a power of e. $e^{x \ln 7}$

In Exercises 3 – 7, simplify the expression using properties of exponents and logarithms.

3. $\ln\left(e^{\tan x}\right)$ $\tan x$

4. $\ln\left(x^2 - 4\right) - \ln(x + 2)$ $\ln(x - 2)$

5. $\log_2\left(8^{x-5}\right)$ $3x - 15$

6. $\dfrac{\left(\log_4 x^{15}\right)}{\left(\log_4 x^{12}\right)}$ $\dfrac{5}{4}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Review Solutions

7. $3\ln x - \ln 3x + \ln(12x^2)$ $\ln(4x^4)$

In Exercises 8 – 10, solve the equation algebraically using logarithms. Give an *exact* answer, such as $\left(\frac{\ln 2}{3}\right)$, and also an approximate answer to the nearest hundredth.

8. $3^x = 19$ $x = \frac{\ln 19}{\ln 3} \approx 2.68$

9. $5^x \ln 5 = 2^x$ $x = \frac{\ln 18 - \ln(\ln 5)}{\ln 5} \approx 1.50$

10. $3^{x+1} = 2^x$ $x = \frac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

What you'll learn about

- Derivative of e^x
- Derivative of a^x
- Derivative of $\ln x$
- Derivative of $\log_a x$
- Extending the Power Rule to arbitrary real powers
- Logarithmic differentiation

… and why

The relationship between exponential and logarithmic functions provides a powerful differentiation tool called logarithmic differentiation.

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Derivative of e^x

If u is a differentiable function of x, then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Example Derivative of e^x

Find $\frac{dy}{dx}$ if $y=e^{x^3}$

$y=e^{x^3}$ where $u=x^3$

$$\frac{dy}{dx}=e^u\frac{du}{dx}$$

$$=e^{x^3}\left(3x^2\right)$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Derivative of a^x

If u is a differentiable function of x and for $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\left(a^u\right) = a^u \ln a \frac{du}{dx}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Derivative of ln x

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Example Derivative of ln x

Find $\frac{dy}{dx}$ if $y=\ln(3x^2)$

$y=\ln(3x^2)$ where $u=3x^2$

$$y' = \frac{1}{3x^2}\frac{d}{dx}(3x^2)$$

$$= \frac{1}{3x^2}(6x)$$

$$= \frac{6x}{3x^2} = \frac{2}{x}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Derivative of $\log_a x$

If u is a differentiable function of x and $u > 0$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Rule 10 Power Rule For Arbitrary Real Powers

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Example Power Rule For Arbitrary Real Powers

If $y=(5x)^{\sqrt[3]{4}}$, find $\frac{dy}{dx}$.

Using the Power Rule

$$\frac{dy}{dx}=\sqrt[3]{4}(5x)^{\left(\sqrt[3]{4}-1\right)}(5)$$

$$=5\sqrt[3]{4}(5x)^{\left(\sqrt[3]{4}-1\right)}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Logarithmic Differentiation

Sometimes the properties of logarithms can be used to simplify the differentiation process, even if logarithms themselves must be introduced as a step in the process.

The process of introducing logarithms before differentiating is called *logarithmic differentiation*.

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Example Logarithmic Differentiation

Find $\frac{dy}{dx}$ for $y = x^{\cos x}$

$y = x^{\cos x}$

$\ln y = \ln x^{\cos x}$ Logs of both sides

$\ln y = \cos x(\ln x)$ Property of logs

$\frac{d}{dx}(\ln y) = \frac{d}{dx}((\cos x)(\ln x))$ Differentiate Implicitly

$\frac{1}{y}\frac{dy}{dx} = (-\sin x)(\ln x) + (\cos x)\left(\frac{1}{x}\right)$ Product Rule

$\frac{dy}{dx} = y\left[(-\sin x)(\ln x) + \frac{\cos x}{x}\right]$

$\frac{dy}{sx} = x^{\cos x}\left[(-\sin x)(\ln x) + \frac{\cos x}{x}\right]$ Substitute for y

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

You may use a graphing calculator to solve these problems.

1. Which of the following gives $\frac{dy}{dx}$ at $x=1$ if $x^3 + 2xy = 9$?

(A) $1\frac{1}{2}$

(B) $\frac{5}{2}$

(C) $\frac{3}{2}$

(D) $-\frac{5}{2}$

(E) $-\frac{11}{2}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

You may use a graphing calculator to solve these problems.

1. Which of the following gives $\frac{dy}{dx}$ at $x=1$ if $x^3 + 2xy = 9$?

(A) $1\frac{1}{2}$

(B) $\frac{5}{2}$

(C) $\frac{3}{2}$

(D) $-\frac{5}{2}$

(E) $-\frac{11}{2}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

2. Which of the following gives $\frac{dy}{dx}$ if $y=\cos^3(3x-2)$?

(A) $-9\cos^2(3x-2)\sin(3x-2)$

(B) $-3\cos^2(3x-2)\sin(3x-2)$

(C) $9\cos^2(3x-2)\sin(3x-2)$

(D) $-9\cos^2(3x-2)$

(E) $-3\cos^2(3x-2)$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

2. Which of the following gives $\frac{dy}{dx}$ if $y=\cos^3(3x-2)$?

(A) $-9\cos^2(3x-2)\sin(3x-2)$

(B) $-3\cos^2(3x-2)\sin(3x-2)$

(C) $9\cos^2(3x-2)\sin(3x-2)$

(D) $-9\cos^2(3x-2)$

(E) $-3\cos^2(3x-2)$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

3. Which of the following gives $\frac{dy}{dx}$ if $y=\sin^{-1}(2x)$?

(A) $-\frac{2}{\sqrt{1-4x^2}}$

(B) $-\frac{1}{\sqrt{1-4x^2}}$

(C) $\frac{2}{\sqrt{1-4x^2}}$

(D) $\frac{1}{\sqrt{1-4x^2}}$

(E) $\frac{2x}{1+4x^2}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Quick Quiz Sections 4.3 – 4.4

3. Which of the following gives $\frac{dy}{dx}$ if $y=\sin^{-1}(2x)$?

(A) $-\frac{2}{\sqrt{1-4x^2}}$

(B) $-\frac{1}{\sqrt{1-4x^2}}$

(C) $\frac{2}{\sqrt{1-4x^2}}$

(D) $\frac{1}{\sqrt{1-4x^2}}$

(E) $\frac{2x}{1+4x^2}$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Chapter Test

1. Find the derivative of $y = \dfrac{2x+1}{2x-1}$.

2. Find the derivative of $r = \ln(\cos^{-1} x)$.

3. Find the derivative of $y = \csc^{-1}(\sec x)$, $0 \le x \le 2\pi$.

4. Find all values of x for which $y = \ln x^2$ is differentiable.

5. Find $\dfrac{d^2y}{dx^2}$ by implicit differentiation given $x^3 + y^3 = 1$.

6. Given $y = 4 + \cot x - 2\csc x$, find an equation for (a) the tangent and (b) the normal to the curve at $x = \pi/2$.

7. Given the function $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$

(a) Is f continuous at $x = 1$?

(b) Is f differentiable at $x = 1$?

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Chapter Test

8. The spread of measels in a certain school is given by $P(t) = \frac{200}{1 + e^{5-t}}$, where t is the number of days since the measels first appeared, and $P(t)$ is the total number of students who have caught the measels to date.

(a) Estimate the initial number of students infected with the measels.

(b) About how many students in all will get the measels?

(c) When will the rate of spread of measels be greatest? What is this rate?

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Chapter Test Solutions

1. Find the derivative of $y = \dfrac{2x+1}{2x-1}$. $dy/dx = \dfrac{-4}{(2x-1)^2}$

2. Find the derivative of $r = \ln(\cos^{-1} x)$. $dr/dx = \dfrac{-1}{\cos^{-1} x\sqrt{1-x^2}}$

3. Find the derivative of $y = \csc^{-1}(\sec x)$, $0 \le x \le 2\pi$.

$$dy/dx = \begin{cases} -1, & 0 \le x < \pi, \ x \ne \pi/2 \\ 1, & \pi < x \le 2\pi, \ x \ne 3\pi/2 \end{cases}$$

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Chapter Test Solutions

4. Find all values of x for which $y = \ln x^2$ is differentiable. All $x \neq 0$

5. Find $\dfrac{d^2y}{dx^2}$ by implicit differentiation given $x^3 + y^3 = 1$. $\dfrac{d^2y}{dx^2} = \dfrac{-2x}{y^5}$

6. Given $y = 4 + \cot x - 2\csc x$, find an equation for (a) the tangent and

(b) the normal to the curve at $x = \pi/2$. (a) $y = -x + \dfrac{\pi}{2} + 2$ (b) $y = x - \dfrac{\pi}{2} + 2$

7. Given the function $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$

(a) Is f continuous at $x = 1$? yes

(b) Is f differentiable at $x = 1$? no

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.

Chapter Test Solutions

8. The spread of measels in a certain school is given by $P(t) = \frac{200}{1 + e^{5-t}}$, where t is the number of days since the measels first appeared, and $P(t)$ is the total number of students who have caught the measels to date.

(a) Estimate the initial number of students infected with the measels. $P(0) \approx 1.339$, so one student.

(b) About how many students in all will get the measels? 200

(c) When will the rate of spread of measels be greatest? What is this rate? After 5 days, with a rate = 50 students/day

Copyright © 2016, 2012, and 2010 Pearson Education, Inc.